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Non-Classical Traveling Solutions in a Nonlinear Klein Gordon Model

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Abstract A model with interparticle inharmonic interaction under the Φ^4 external potential is studied. Making use of the special relation of relevant parameters, in the continuous limit, it was possible to obtain various non-classical soliton solutions, specifically, compact and peak-like solutions. The solutions undergo a jump on their first derivatives at some points of the space-time manifold. These analytic solutions were obtained by considering strong restrictions on their velocities and on the jump conditions. The energy concentrated in these solutions shows in several cases a discrete spectrum.

Keywords Non-classical solitons · Peakons · Compactons · Compact bubbles · Discrete energies

1 Introduction

The investigations for obtaining non classical solutions like peakons, compactons or cusps are still rich fields of research. The concept of compacton, appeared first in the context of solutions of nonlinear lattices, before they were discovered in some other branches of physics. Rosenau and Hyman [19] studied the special type of the modified Korteweg De Vries equation $KdV(m,n)$ and found that the solitary waves might compactify in the presence of nonlinear dispersion. The compact soliton due to their absence of tails will not interact with other structures at long distances. In opposite, the classical soliton, for instance, in nonlinear

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optical fibers, generates strong restriction limitations during the transmission of information between two points at long distances [1]. Other areas of applications are the formation of vortices in hydrodynamics and the formation of dislocations in solid bodies under stress and self-capture of energy in the proteins (self-trapping) [18]. This type of compact solutions could better model the matter without tails and consequently have direct applications in the study of particles because of their local interaction properties [4]. Kivshar [16] discovered the possibility of the existence of compact breathers in a system of identical particles with pure pair inharmonic interaction and further P. Tchofo and M. Remoissenet found compact breathers for the same system that we are studying in this work. In the article [3] the compact like solutions were discovered for the equation that describe the shallow water. The investigation of compactons in a chain of dispersively nonlinear coupled oscillators was done by Pikovsky and Rosenau [17]. The relevance of compactons for applications in practical and theoretical studies should be found in the works [2, 7, 10, 12, 15, 17, 20] and citation therein. We quoted only several papers and obviously it is not complete.

On the other hand, Camassa and Holm [5] (CH), began a systematic study for water waves and found piecewise analytic solutions that were named as peakons. These analytic solutions usually have a corner at their crest. Further Zheng and Lai studied the generalized Camassa-Holm equation and found peak patterns under various circumstances [23]. The generalization of CH system associated with singular solutions can be found in the work of Tian and Sun [22]. Various families of higher order peak solutions are also found in the Kronig Penney model in the presence of the uniform cubic quintic nonlinearity [11]. Recently, like the Camassa-Holm and Degasperis-Procesi equations, the Novikov partial differential equation with cubic nonlinearity also supports peakons [14]. The two-component analog of CH equation with respective solutions could be found in the work [13] and citation therein.

The system studied here consist of a chain of particles joined along the line with anharmonic inter particle potential and subjected to an external θ^4 potential. We study this system in the continuum limit and were able to find compact and peak solutions. So, we focus on the study of analytical solutions of the following equation:

$$\ddot{\theta} - \kappa\left[3\left(\frac{\partial\theta}{\partial x}\right)^2\frac{\partial^2\theta}{\partial x^2}\right] + (\alpha\theta - \beta\theta^3) = 0. \tag{1}$$

This equation involves one potential between neighbors (second term) and the potential on site with inharmonic interactions (last terms). In several works [6, 8, 9, 21], it has been demonstrated that this type of system with similar characteristics for the Hamiltonian (4) supports interesting non classical solutions.

Having considered the two types of boundary conditions, we were able to obtain solutions with and without tails. Since our solutions were piece wise analytical we used the jump conditions for obtaining physical appropriated solutions. The jump conditions were verified and helped us for selecting the physical accepted solutions. After this treatment, the energy of the obtained structures were found. The analysis done on the energetic spectrum showed unexpected discrete values for determining the energy of the solutions.

The next section gives the derivation of the motion equation from the hamiltonian model. Section 3 deals with the obtention of traveling structures linked to the analysis of jump conditions. Our solutions are mainly obtained by gluing different parts of exact solutions of transcendental algebraic equations. The non-classical traveling waves are exposed widely in this section taking in consideration two boundary conditions: the trivial and the condensate ones. In Sect. 4 we show the appearance of discrete energy spectrum for the solutions. Finally we conclude the work with some comments that is done in Sect. 5.

2 Mathematical Model and Equations of Motion

Our studies are realized in a one dimensional space time manifold of the nonlinear Klein Gordon model. Consider a lattice of identical atoms for which the neighboring interactions are modeled by the pure nonlinear potential

$$U(\theta_n - \theta_{n-1}) = \frac{1}{4}\kappa(\theta_n - \theta_{n-1})^4 \tag{2}$$

Where $\theta(t)$ is the on-site degree of freedom and κ is a parameter that controls the forces between the nonlinear pair of pendulum. This potential represents the interaction in a system of identical atoms through inharmonic forces. Next, consider the system is subjected to external forces given by the potential V at the site "n"

$$V(\theta_n) = \frac{\alpha\theta_n^2}{2} - \frac{\beta\theta_n^4}{4} \tag{3}$$

This term represents the potential for external interaction with the neighborhood and possesses two equilibrium positions like a pendulum. Thus, the Hamiltonian of the model can be written as

$$H = \sum_{n=1}^{\infty}\left[\frac{1}{2}\dot{\theta}_n^2 + U(\theta_n - \theta_{n-1}) + V(\theta_n)\right] \tag{4}$$

Additionally, the potential part $V(\theta)$ has the characteristic that it can confine the solution inside its infinite walls. As it is easy to see this potential has a harmonic contribution in the first term and inharmonic contribution in the second one. The parameter β is the measure of the harmonicity and plays an important rule in the stabilization.

The corresponding equation of motion of the nth particle becomes

$$\frac{d^2\theta_n}{dt^2} - \frac{dU(\theta_{n+1} - \theta_n)}{d\theta_n} - \frac{dU(\theta_n - \theta_{n-1})}{d\theta_n} = -\frac{dV(\theta_n)}{d\theta_n} \tag{5}$$

Where as we pointed out $U(\theta_n - \theta_{n-1})$ describes an interaction potential between pair of neighboring and $V(\theta_n)$ describes a site potential of interaction with the external medium. Substituting the potentials in the equation of motion (5) one obtains the nonlinear lattice equation

$$\ddot{\theta}_n + \kappa[(\theta_n - \theta_{n-1})^3 + (\theta_n - \theta_{n+1})^3] + (\alpha\theta_n - \beta\theta_n^3) = 0 \tag{6}$$

Suppose that the field θ varies slowly from one site to another, and $2\beta \ll \kappa$, thus one can use the continuum approximation that was successfully employed in [9, 21] and (6) transforms to the next nonlinear differential equation

$$\ddot{\theta} - \kappa\left[3\left(\frac{\partial\theta}{\partial x}\right)^2\frac{\partial^2\theta}{\partial x^2}\right] + (\alpha\theta - \beta\theta^3) = 0 \tag{7}$$

Let us analyze the important case of traveling waves as solutions of the nonlinear differential equation (7). Thus the new variables are

$$s = x - ut \quad \rightarrow \quad \theta(s) = \theta(x - ut) \tag{8}$$

with arbitrary values of velocities u. Making the standard change of variables for traveling waves in (7) we obtain the new nonlinear equation of motion

$$u^2\frac{d^2\theta}{ds^2}-\kappa\left[3\left(\frac{d\theta}{ds}\right)^2\frac{d^2\theta}{ds^2}\right]+(\alpha\theta-\beta\theta^3)=0 \tag{9}$$

We now multiply (9) with $\frac{d\theta}{ds}$ and integrate again, to obtain the next ordinary differential equation

$$\left(\frac{d\theta}{ds}\right)^4-a\left(\frac{d\theta}{ds}\right)^2+b(\alpha-\beta\theta^2)^2=0 \tag{10}$$

with the simplified parameters

$$a=\frac{2u^2}{3\kappa},\qquad b=\frac{1}{3\beta\kappa} \tag{11}$$

3 Non Classical Traveling Waves

3.1 Jump Conditions

The jump condition was used as an approximative tool for building soliton like solutions. This method is used worldwide in the theory of complex liquids and other branches of physics. So, we use this approach for obtaining solutions $u(x)$ by gluing available branches. Assume that at some critical point s_0 occur the jump i.e. the first derivatives of the branches satisfy the equation

$$\lim_{\varepsilon\to 0}\frac{\partial u}{\partial x}\bigg|_{x_0-\varepsilon}^{x_0+\varepsilon}\neq 0 \tag{12}$$

This expression would show the behavior of the derivatives when the solution crosses the point x_0. If the jump is different from zero then the derivative is discontinuous and the jump occurs. This would enable us with some tool to glue the branches. First of all we integrate the equation in the neighborhood of the point s_0 under which the branches are glued

$$\int_{s_0-\varepsilon}^{s_0+\varepsilon}\left(u^2\frac{d^2\theta}{ds^2}-\kappa\left[3\left(\frac{d\theta}{ds}\right)^2\frac{d^2\theta}{ds^2}\right]+(\alpha\theta-\beta\theta^3)\right)ds=0 \tag{13}$$

After subsequent calculations we derive the condition that has to be satisfied by the branches at the point of jump

$$\theta_s(s_0)=\pm\sqrt{\frac{a}{2}} \tag{14}$$

This result will be used in the next sections in different situations concerning the building of solutions.

The jump conditions (14) permit us to obtain physically accepted solutions among the all possible mathematical solutions. All the solutions obtained below are inverse functions of the function $s(\theta)$. These solutions are formed by gluing at crucial points different branches that satisfy the initial and jump conditions. In several cases, for each solution its corresponding anti-soliton partner could be composed by inverting the sign of the branches that form the solution. Only at a few number of them the continuity of the first derivation is satisfied. In few cases it is possible to conserve the continuity of the solution. In some other ones when discontinuities occur the branches are glued each other by translations. This translation does not modify the properties of the solution since each part travels with the same velocity. The necessary step Δ for gluing the solutions will be defined by the boundary condition s_0 and corresponds to some retarded value in the space s. The value of s_0 will be determined by the point θ_0 as initial condition. When the boundary condition is of the type of condensed one, the values of s do not always be the same, that can be denoted as $\theta_0(s)^{-1}$.

Next, we impose physical boundary conditions for solving (10). These conditions will provide us with the specific restrictions to the values of the wave velocities and to the parametric domains of the model. First, we take into account the trivial boundary conditions and subsequently we will use the condensate type of boundary conditions.

3.2 Trivial Boundary Conditions

We analyze the nonlinear equation (10) subjected to the boundary condition of the following type $s \to \pm\infty$, $\theta \to 0$, $\theta_s \to 0$. From here it is easy to notice that the only possible solutions are those for which the coupling α vanishes i.e. $\alpha = 0$. Consequently, (10) is transformed to the next one:

$$\left(\frac{d\theta}{ds}\right)^4 - a\left(\frac{d\theta}{ds}\right)^2 + b\beta^2\theta^4 = 0 \tag{15}$$

Resolving (15) by quadratures we obtain

$$\xi\sqrt{\frac{a}{2}}ds = \frac{d\theta}{\sqrt{1+\mu\sqrt{1-\sigma\theta^4}}}do \tag{16}$$

where the parameter σ is defined by

$$\sigma = \frac{3\kappa\beta}{u^4} \tag{17}$$

Because of the signs of $\xi = \pm 1$ and $\mu = \pm 1$, the expression (16) permits us to consider four different cases for the branches for composing solutions and we denote them as: $({}^{\mu}s_{\xi})$. The replacement of the jump condition (14) into (15) yield the field values where jumps occur at.

$$\theta_0 = \pm\sqrt[4]{\frac{1}{\sigma}} \tag{18}$$

These values are used thus for selecting the acceptable branches and we are now in position to obtain the following solutions:

Compact Solution The first case is defined by: $\mu = +1$ and $\xi = \pm 1$. Taking in mind these signs we integrate (16). By solving this equation we obtain two analytical implicit solutions that we denoted as: ${}^{+}s_{+}$ y ${}^{+}s_{-}$ in accordance with the signs of μ y ξ.

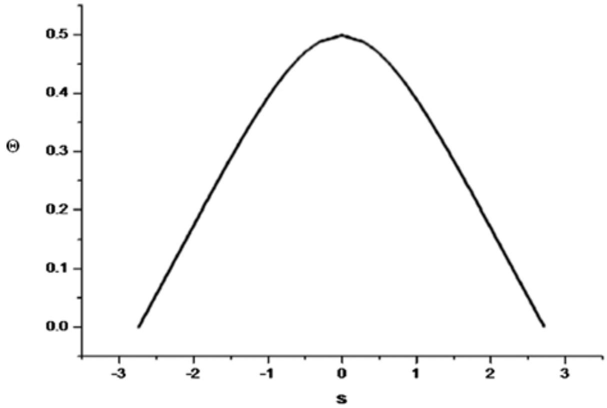

Fig. 1 The profile of compact pulse where for qualitative reasons we used the values $a = 1$ and $\sigma = 16$

$$\xi\sqrt{\frac{a\sqrt{\sigma}}{2}}({}^{+}s - s_0) = \mathrm{Arc}\tan\left[-1+\sqrt{\frac{2\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}}\right] + \mathrm{Arc}\tan\left[1+\sqrt{\frac{2\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}}\right]$$
$$-\frac{1}{4}\mathrm{Ln}\left[\left(-1+\sqrt{\frac{2\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}} - \frac{\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}\right)^2\right]$$
$$+\frac{1}{4}\mathrm{Ln}\left[\left(1+\sqrt{\frac{2\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}} + \frac{\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}\right)^2\right]$$
$$-\sqrt{\frac{2\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}}$$

The first branch is defined by $\theta({}^{+}s_{+})$ in the interval $s \in (-\infty, 0]$ and the second one as $\theta({}^{+}s_{-})$ in the interval $s \in [0, \infty)$. Using these branches it is composed a structure in the form of compact pulse with bell shape similar to orthodox solitons but without tails. The profile of compact pulse is shown in Fig. 1.

Peak Solution In the second case the solutions are defined by the signs $\mu = -1$ y $\xi = \pm 1$ that are applied also to (16). Under these circumstances we denote the new branches by: ${}^{-}s_{+}$ y ${}^{-}s_{-}$ in accordance with the signs of ξ and μ.

$$\xi\sqrt{\frac{a\sqrt{\sigma}}{2}}({}^{-}s - s_0) = -\mathrm{Arc}\tan\left[-1+\sqrt{\frac{2\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}}\right] - \mathrm{Arc}\tan\left[1+\sqrt{\frac{2\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}}\right]$$
$$-\frac{1}{4}\mathrm{Ln}\left[\left(-1+\sqrt{\frac{2\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}} - \frac{\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}\right)^2\right]$$
$$+\frac{1}{4}\mathrm{Ln}\left[\left(1+\sqrt{\frac{2\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}} + \frac{\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}\right)^2\right]$$
$$-4\sqrt{\frac{1+\sqrt{1-\sigma\theta^4}}{2\sqrt{\sigma}\theta^2}}$$

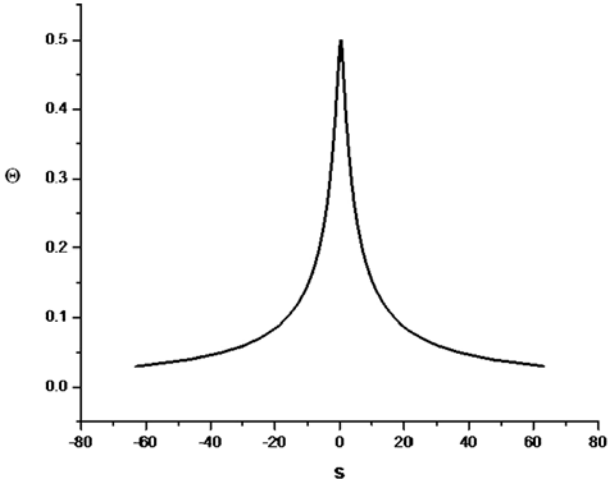

Fig. 2 Typical peak soliton like solution with parameter values: $a = 1$ and $\sigma = 16$

The first branch $\theta(^{-}s_{+})$ of the structure is defined on the interval $s \in (-\infty, 0]$, and the second one $\theta(^{-}s_{-})$ on the interval $s \in [0, \infty]$. From these branches it is composed a non-classical solution that is known as "peakon". This solution has a discontinuity of the first derivative in the apex and its profile is shown in Fig. 2. The other two cases do not satisfy the trivial boundary conditions and consequently we disregard them.

3.3 Condensate Boundary Conditions

In this part we analyze the condensate or nontrivial boundary conditions for (10) under which the field cannot reach the zero values at infinities, instead it takes a constant value at the frontiers: $s \to \pm\infty$, $\theta \to \theta_0 = \text{const.}$, $\theta_s \to 0$. Under these conditions the new nonlinear equation is reduced to

$$\left(\frac{d\theta}{ds}\right)^4 - a\left(\frac{d\theta}{ds}\right)^2 + b(\alpha - \beta\theta^2)^2 = 0 \tag{19}$$

and the limiting field values at infinities are:

$$\theta_0 = \pm\sqrt{\frac{\alpha}{\beta}} \tag{20}$$

These values correspond to minimums of the double degenerated potential. Resolving (19) by quadratures we obtain the expression

$$\pm\sqrt{2fa}ds = \sqrt{\frac{1}{1 \mp \sin z}}\frac{\cos z dz}{\sqrt{1 \pm \cos z}} \tag{21}$$

where the following relations have been used:

$$\pm\sin z = 1 - f\theta^2, \quad \alpha = \sqrt{\frac{\beta u^4}{3\kappa}}, \quad f = \frac{\beta}{\alpha} \tag{22}$$

and $\sigma\alpha = 1$.

Now we have four particular expressions that are distinguished by the signs of the combination denoted as ${}^{\mp\pm}s$. For better treatment of the jump condition we rewrite (19) in the following manner

$$\frac{d\theta}{ds} = \pm\sqrt{\frac{a}{2}}\sqrt{1 \pm \sqrt{1-(1-f\theta^2)^2}} \tag{23}$$

and try to guess the value for θ at the jump. When we substitute (14) in the previous equation, we find the field points at which the jumps occurs

$$\theta_0 = \pm\sqrt{\frac{2}{f}} \quad \text{and} \quad \theta_0 = 0 \tag{24}$$

These values will be used a posteriori for building acceptable solutions. Integrating (21) it is obtained for the general case a family of non-classical solutions for the inverse function $s = s(\theta)$. Among the all possibilities we report here only those solutions that represent physical interest. From (21) one can obtain transcendental equations for the unknown field function $\theta(s)$. Thus for identifying the solutions we use the notation ${}^{pq}s_{\varsigma\omega}$, being $\varsigma = \omega = p = q = \pm 1$.

3.3.1 Case I

For this case there are four pairs of branches that is determined by the transcendental equation obtained by integrating (21)

$$\varsigma 2\sqrt{fa}({}^{-+}s_{\varsigma\omega} - s_0) = -\operatorname{Arc\,sin}(1-f\theta^2) + \operatorname{Ln}\left(\frac{1+\omega\theta\sqrt{2f-f^2\theta^2}}{2}\right) \tag{25}$$

where

$$f = \frac{\beta}{\alpha}, \qquad a = \frac{2u^2}{3\kappa} \tag{26}$$

As it can be seen from the transcendental equations in general, the branches for determining solutions depend on the signs of ς, ω and on the parameter values. We present here the solutions that have been built by joining the branches for different values of the signs ς, ω with $p = -1$, $q = +1$ and organize them in the following manner

Solution A For this solution the initial condition was taken as $\theta_0 = 0$. The first branch is defined as: $\theta({}^{-+}s_{++})$ in the interval of $s \in (-\infty, 0]$ and the second branch as $\theta({}^{-+}s_{--})$ in the interval $s \in [0, \infty)$. Similar structures were found in [9] and [2], and is known as compact kink. This solution looks like the famous kink solution but it is remarkable to observe that this solution does not have wings (Fig. 3). It is a twist in the independent variable and represents not only a physical interest but also topological meaning.

Solution B In this case, the first branch of the structure is defined as: $\theta({}^{-+}s_{+-})$ in the interval $s \in (-\infty, 0]$ and the second branch as: $\theta({}^{-+}s_{--})$ in the interval of $s \in [0, \infty)$. These branches permit to build a structure like a compact-pulse a localized structure on the condensate with a popular bell shape. This solution is also named as soliton on the step (Fig. 4).

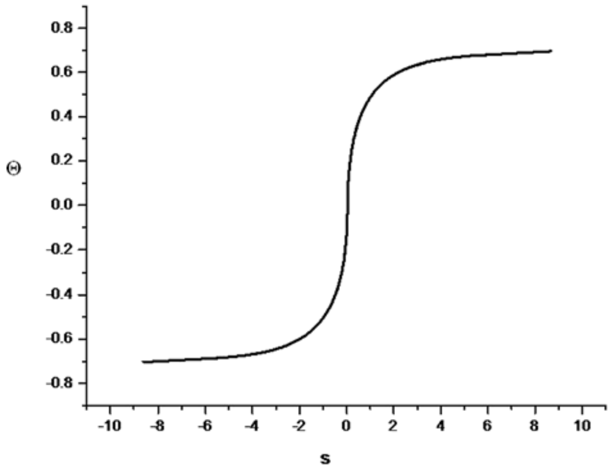

Fig. 3 Compact kink structure that is localized inside the interval defined by the equations ${}^{-+}s_{++}$ and ${}^{-+}s_{--}$

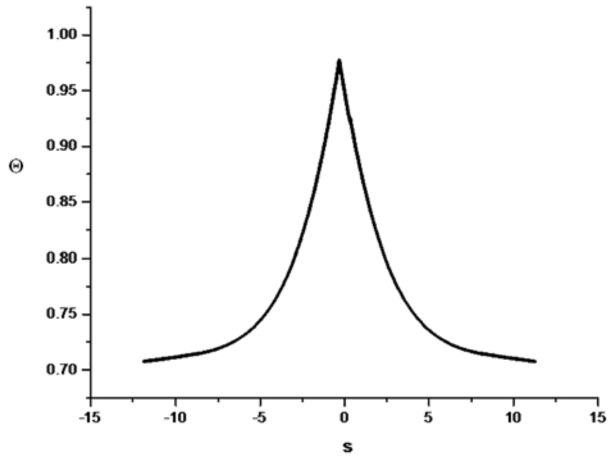

Fig. 4 Soliton like structure on the step, with bell shape similar to the common soliton solution

3.3.2 Case II

The next interesting cases arise from the following transcendental equation:

$$2\varsigma\sqrt{fa}({}^{++}s - s_0) = -\operatorname{Arc}\sin(1 - f\theta^2) - \operatorname{Ln}\left(\frac{1 + \omega\theta\sqrt{2f - f^2\theta^2}}{2}\right) \tag{27}$$

As in the previous important cases, each branch is denoted by ${}^{++}s_{\varsigma\omega}$. We thus organize the following solutions as

Solution C The first branch of the structure is defined as: $\theta({}^{++}s_{-+})$ in the interval $s \in (-\infty, 0]$ and for the second branch $\theta({}^{++}s_{++})$ in the interval $s \in [0, \infty)$. From these pieces it has been composed a kink structure. The initial condition used $\theta_0 = 0$ allows one to match the jump condition as before (Fig. 5).

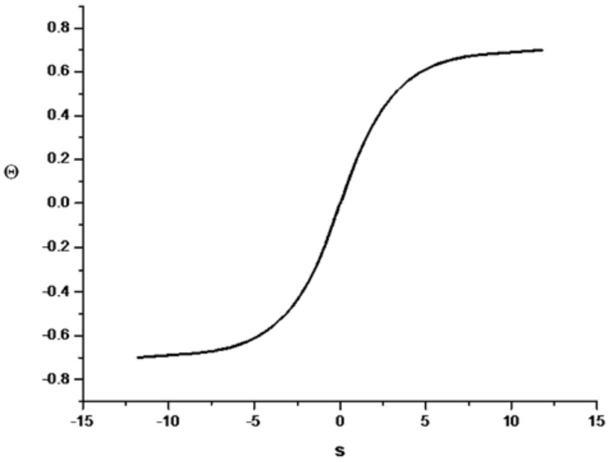

Fig. 5 Kink like structure defined by the equations ${}^{++}s_{-+}$ y ${}^{++}s_{++}$

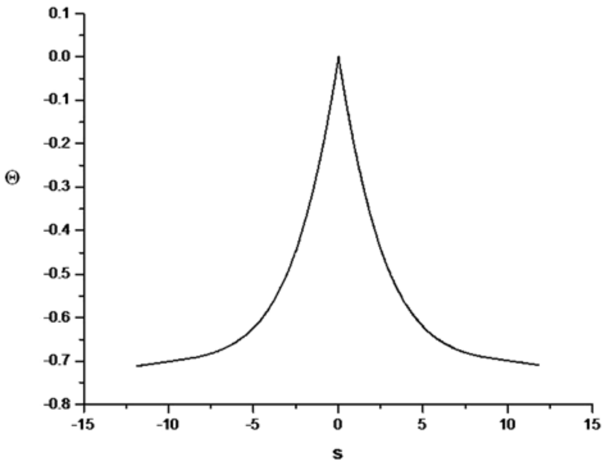

Fig. 6 Anti-bubble compact structure, that is characterized by the absence of wings and is the mirror image of the normal compact bubble solution

3.3.3 Case III

The equation for determining the branches is

$$2\varsigma\sqrt{fa}({}^{+-}s - s_0) = -\operatorname{Arc}\sin(1 - f\theta^2) + \operatorname{Ln}\left(\frac{\sqrt{f}\theta + \sqrt{2 - f\theta^2}}{2}\right)^2 \tag{28}$$

Solution D These branches we designate as ${}^{+-}s_\varsigma$. In this occasion we were able to find a solution named as anti-bubble compact solution because the field takes only negative values. In other words it is a mirror image of a compact bubble solution and is depicted in Fig. 6.

4 Energy Properties

First we deal with solutions that arise applying a trivial boundary condition. The energy of each non-classical solution is obtained by using the Hamiltonian density:

$$H = \frac{u^2}{2}\left(\frac{d\theta}{ds}\right)^2 + \frac{1}{4}\kappa\left(\frac{d\theta}{ds}\right)^4 - \frac{\beta\theta^4}{4}$$

Consequently, for the both first solutions the energy can be obtained from the following expression:

$$\begin{aligned} E = \sqrt{\frac{a}{2}} \int_{\theta_1}^{\theta_2} \frac{u^2}{2}\sqrt{1 \pm \sqrt{1-\sigma\theta^4}} + \frac{a}{8}\kappa\left(\sqrt{1 \pm \sqrt{1-\sigma\theta^4}}\right)^3 \\ - \frac{\frac{\beta\theta^4}{4}}{\frac{a}{2}\sqrt{1 \pm \sqrt{1-\sigma\theta^4}}} d\theta \end{aligned} \tag{29}$$

where the inferior limit corresponds to the height of the pulse or peakon, i.e. the point where the branches are gluing and represents the height of the wave, while the superior limit is given by the trivial boundary conditions, thus we have:

$$\theta_1 = \pm\sqrt[4]{\frac{1}{\sigma}}, \qquad \theta_2 = 0$$

After making some change of variables and integrating (29) one can compute the energy of the first compact pulse solution:

$$\begin{aligned} E_1 = \frac{u^2\sqrt{a}}{\sigma^{1/4}}\left[-\frac{63}{192}(4p+1)\pi + \frac{63}{192}\mathrm{Ln}(\sqrt{2}+1)^2 + \frac{1}{8}\left(\frac{17}{2\sqrt{2}} - \sin\frac{3(4q+1)\pi}{4}\right)\right] \\ + \frac{u^2\sqrt{a}}{\sigma^{1/4}}\left[\frac{63}{192}r\pi\right] \end{aligned}$$

Where the parameters p, q, r take only integer values. Consequently, the energetic values of the compact pulse solution present an unexpected behavior showing a discrete spectrum.

Similarly, for the peak solution the energy values are given by

$$\begin{aligned} E_2 = \frac{u^2\sqrt{a}}{\sigma^{1/4}}\left[\frac{63}{96}p\pi + \frac{63}{96}\mathrm{Ln}(\sqrt{2}-1) + \frac{1}{8}\left(\frac{3}{\sqrt{2}} + \sin\frac{3(4q+1)\pi}{4}\right)\right] \\ - \frac{u^2\sqrt{a}}{\sigma^{1/4}}\left[\frac{63}{192}r\pi\right] \end{aligned}$$

Also in this case we obtained a discrete spectrum of energy because of the values of p, q, r. We observe that these non-classical solutions support discrete energy values and in some sense they have certain similarities with quantum properties. It is expected that in this kind of one dimension or quasi one dimension model, should be easily excited those solutions with the lowest energy values.

For the second type of solutions with nontrivial boundary conditions it is possible to obtain their energies by using the Hamiltonian density:

$$H = \frac{1}{2}\left(u\frac{d\theta}{ds}\right)^2 + \frac{1}{4}\kappa\left(\frac{d\theta}{ds}\right)^4 + \left(\frac{\alpha\theta^2}{2} - \frac{\beta\theta^4}{4}\right)$$

From which the energy spectrum will be calculated using in standard approach and we obtain:

$$E = \int_{\theta_1}^{\theta_2} \frac{u^2}{2}\left(\sqrt{\frac{a}{2}}\sqrt{1 \pm \sqrt{1-(1-f\theta^2)^2}}\right) + \frac{1}{4}\kappa\left(\sqrt{\frac{a}{2}}\sqrt{1 \pm \sqrt{1-(1-f\theta^2)^2}}\right)^3 + \frac{\sqrt{\frac{2}{a}}(\frac{\alpha\theta^2}{2} - \frac{\beta\theta^4}{4})}{\sqrt{1 \pm \sqrt{1-(1-f\theta^2)^2}}} d\theta \tag{30}$$

Where we take into consideration the following relations:

$$\frac{a}{2}\kappa = \frac{u^2}{2}, \qquad \alpha = \sqrt{\frac{\beta u^4}{3\kappa}}, \qquad \frac{\alpha}{a}\frac{1}{f} = \frac{u^2}{2}, \qquad \frac{\beta}{2a}\frac{1}{f^2} = \frac{u^2}{4}$$

After using change of variables and the parameter values for each cases, we can integrate the expression (30) and evaluate the energetic values. Below we present the results, where the subindex n of the expression E_n corresponds to the type of solutions with the nontrivial boundary condition presented in the previous section

$$E_A = \frac{u^2}{8}\sqrt{\frac{a}{f}}\left[-\frac{65}{4} - \frac{13\pi}{16}[(m+m') - 4(r+r')] + 3\,\mathrm{Ln}\,\frac{1}{2}\right]$$

$$E_B = \frac{u^2}{8}\sqrt{\frac{a}{f}}\left[\frac{23}{4} + \frac{13\pi}{16}[4(s+s') - (l+l')] + 3\,\mathrm{Ln}\,\frac{1}{2}\right]$$

$$E_C = -\frac{u^2}{8}\sqrt{\frac{a}{f}}\left\{\frac{3}{2} + \frac{13}{16}\pi(s+s') + \frac{3}{16}[(-1)^s + (-1)^{s'}] - \left[-3 + \frac{13}{16}\left(\pi(m+m') + \cos\frac{m\pi}{4} + \sin\frac{m\pi}{4} + \cos\frac{m'}{4} + \sin\frac{m'\pi}{4}\right) + \mathrm{Ln}\,\frac{1}{2}\right]\right\}$$

$$E_D = -\frac{u^2}{8}\sqrt{\frac{a}{f}}\left\{\frac{15}{8} - \frac{13\pi}{4}(p+p'+1) - \left[\frac{21}{8} - \frac{13\pi}{16}(m-m') + \mathrm{Ln}\,\frac{1}{2}\right]\right\}$$

As can be seen from the previous relations all these energies have specific discrete values because of the integer values of $m, m', l, l', p, p', r, r'$. The general form of the energy for both solutions with trivial boundary condition has the following form

$$E = \rho u^4\left(\frac{4}{(3\kappa)^3\beta}\right)^{1/4}$$

where ρ is the numerical discrete coefficient. Comparing both energies in their minimal values it is observed that the peak like solution has less energy than the compact pulse. Thus we can say that the peakon is the excitation with major probability to emerge along the chain.

For the case of solutions with nontrivial boundary condition the general form of the energies is given in terms of the parameters u, κ and β:

$$E = \gamma\frac{u^4}{8}\left(2\kappa^3\beta\right)^{-1/4}$$

Where γ is the numerical coefficient for each solution and u is determined by α, κ y β. Similarly, by analyzing the minimal energy values and comparing the obtained results we can infer that when the condensate boundary condition is applied, the excitation with more probabilities to appear is a compact pulse structure.

5 Conclusions

For the system of coupled particles in the discrete system, it has been used potentials of inter-particle interactions of anharmonic type and the substrate four potential ϕ^4. This system was transformed to a continuous model which has been resolved for the traveling wave solutions. The anharmonicity or the nonlinear dispersion shows a rich variety of new possible type of static or traveling solutions. It has been considered two types of boundary conditions: the trivial and the condensed boundary ones. The sign of the an-harmonic term determines the attractive or repulsive type of interactions between particles. The solutions were obtained from six different transcendental equations. The first two of them were used for the trivial boundary conditions and the next four equations were analyzed with the condensed type of boundary conditions. As it is well known the solitons are special solutions. In this work we have found strong restrictions for the velocities of each composed solution in dependence on the specific parameter values. In addition it has been obtained the corresponding energies of each solution. The compact pulse and the peak like solution could coexist in the same parameter regions. For the condensed type of boundary condition it has been found the following structures: compact-kinks, kinks, solitons on step and anti bubble compact solutions. In this case, all solutions exist for the same region of the principal parameters. In other words, all solutions can coexists at the same time if they are excited in the system. These solutions complements the rich variety of solutions reported in the literature.

All the energies of our structures have discrete values. These values are determined by integer numbers similar to solutions of vibrating strings in the classical example of oscillations in linear systems. The velocities are constrained and we can identify them because of their dependencies on the parameters κ, α, β given by the potential pieces of the corresponding hamiltonian. For the solutions with condensed boundary conditions we have

$$u = \sqrt[4]{\frac{3\alpha^2\kappa}{\beta}}$$

For the case of solutions with trivial boundary conditions, the velocity satisfies the relation

$$u = \sqrt[4]{\frac{3\kappa\beta}{\sigma}}$$

As it can be seen from the expressions for velocities, these quantities are not arbitrary. They are restricted by the values of the parametric domain of concrete physical situation. It in addition could enable one with some tool for manipulating the nonlinear waves in real experiments.

The nonlinear classical structures that have great chance to be excited in the first case is the peakon and in the second one is the compact pulse structure. The nonlinear system analyzed here possesses a certain selective property. Having defined the parameters of the model, along the chain will be excited only nonlinear traveling structures with specific values of discrete energies and velocities. The results of this work could be used in continuum versions of discrete models like in DNA or crystals for example.

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